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Distributivity numbers of $\mathcal{P}(\omega)/\text{fin}$ and its friends

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Abstract

This brief survey on distributivity numbers is an exposition of the talk which I gave at RIMS in October 2005.

1 Distributivity numbers of Boolean algebras

Let \mathbb{P} be a separative partial order. $D \subseteq \mathbb{P}$ is *dense* if for all $p \in \mathbb{P}$ there is $q \leq p$ with $q \in D$. D is *open* if for all $p \in D$, any $q \leq p$ belongs to D . The *distributivity number* (or *height*) of \mathbb{P} , $\mathfrak{h}(\mathbb{P})$, is the least size of a family \mathcal{D} of open dense subsets of \mathbb{P} such that $\bigcap \mathcal{D}$ is not dense. Note that $\bigcap \mathcal{D}$ necessarily is open. Equivalently, $\mathfrak{h}(\mathbb{P})$ is the least size of a family \mathcal{A} of maximal antichains of \mathbb{P} which has no common refinement. Here, for maximal antichains $A, B \subseteq \mathbb{P}$, we say that A *refines* B if for all $p \in A$ there is $q \in B$ with $p \leq q$. If \mathbb{A} is an atomless Boolean algebra, we let $\mathbb{A}^+ = \mathbb{A} \setminus \{0\}$ denote the partial order of its positive elements and define $\mathfrak{h}(\mathbb{A}) := \mathfrak{h}(\mathbb{A}^+)$. Similarly for other cardinals.

Fact 1. $\mathfrak{h}(\mathbb{P})$ is a regular cardinal. \square

$\mathfrak{h}(\mathbb{P})$ is an invariant of \mathbb{P} as a forcing notion, that is, it does not depend on the particular realization of \mathbb{P} . Equivalently, $\mathfrak{h}(\mathbb{P}) = \mathfrak{h}(\text{r.o.}(\mathbb{P}))$ where $\text{r.o.}(\mathbb{P})$ is the *completion* of \mathbb{P} , i.e., the unique complete Boolean algebra forcing equivalent with \mathbb{P} . For a topological space X , $\text{r.o.}(X)$ is the *Boolean algebra of regular open subsets* of X where $O \subseteq X$ is called *regular open* if it is open and $\text{Int}(\text{Cl}(O)) = O$. It is well-known that $\text{r.o.}(X)$ is a complete Boolean algebra. If $X = \mathbb{P}$ with the topology introduced above, the mapping $p \mapsto O_p = \{q \in \mathbb{P} : q \leq p\}$ is a *dense embedding* of \mathbb{P} into $\text{r.o.}(\mathbb{P})$ and thus \mathbb{P} and $\text{r.o.}(\mathbb{P})$ are indeed forcing

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equivalent. In case \mathbb{A} is an atomless Boolean algebra, there is an alternative description of $\text{r.o.}(\mathbb{A}) := \text{r.o.}(\mathbb{A}^+)$: namely, if $\text{St}(\mathbb{A})$ is the Stone space of \mathbb{A} , then $\text{r.o.}(\mathbb{A}) = \text{r.o.}(\text{St}(\mathbb{A}))$.

From the forcing-theoretic point of view, $\mathfrak{h}(\mathbb{P})$ is the minimal cardinal κ such that there are $p \in \mathbb{P}$ and a \mathbb{P} -name \dot{f} for a function from κ to the ground model V such that $p \Vdash_{\mathbb{P}} \dot{f} \notin V$. Indeed if $\lambda < \mathfrak{h}(\mathbb{P})$ and $\dot{f} : \lambda \rightarrow V$, then, letting D_α , $\alpha < \lambda$, be the open dense subset of \mathbb{P} consisting of conditions which decide the value $\dot{f}(\alpha)$, $D = \bigcap_{\alpha < \lambda} D_\alpha$ is dense and for any $p \in D$ there is $f_p \in V$ such that $\Vdash_{\mathbb{P}} \dot{f} = f_p$. Thus $\Vdash_{\mathbb{P}} \dot{f} \in V$. On the other hand, if D_α , $\alpha < \mathfrak{h}(\mathbb{P})$, are open dense such that $D = \bigcap_{\alpha < \mathfrak{h}(\mathbb{P})} D_\alpha$ is not dense and $A_\alpha = \{p_{\alpha, \gamma} : \gamma < \kappa_\alpha\} \subseteq D_\alpha$ are maximal antichains, then, letting $\dot{f} : \mathfrak{h}(\mathbb{P}) \rightarrow V$ be the \mathbb{P} -name defined by $p_{\alpha, \gamma} \Vdash_{\mathbb{P}} \dot{f}(\alpha) = \gamma$, we see that if $p \in \mathbb{P}$ is such that no $q \leq p$ belongs to D then $p \Vdash_{\mathbb{P}} \dot{f} \notin V$.

If \mathbb{P} is *homogeneous*, that is, if O_p is forcing equivalent with \mathbb{P} for all $p \in \mathbb{P}$ (equivalently, if $\text{r.o.}(\mathbb{P}) = \text{r.o.}(O_p)$ for all $p \in \mathbb{P}$), then $\mathfrak{h}(\mathbb{P})$ is the least size of a family \mathcal{D} of open dense subsets of \mathbb{P} with $\bigcap \mathcal{D} = \emptyset$. Equivalently, $\mathfrak{h}(\mathbb{P})$ is the least κ such that $\Vdash_{\mathbb{P}} \dot{f} \notin V$ for some \mathbb{P} -name $\dot{f} : \kappa \rightarrow V$.

We write $\mathbb{P} <_{\circ} \mathbb{Q}$ if there is a *complete embedding* from \mathbb{P} into \mathbb{Q} .

Fact 2. If $\mathbb{P} <_{\circ} \mathbb{Q}$ then $\mathfrak{h}(\mathbb{P}) \geq \mathfrak{h}(\mathbb{Q})$. \square

We briefly mention two cardinals which are closely related to $\mathfrak{h}(\mathbb{P})$. A *tower* $T \subseteq \mathbb{P}$ is a well-ordered decreasing chain without a lower bound. The *tower number* $\mathfrak{t}(\mathbb{P})$ of \mathbb{P} is the least size of a tower in \mathbb{P} . $q \in \mathbb{P}$ *splits* $p \in \mathbb{P}$ if p and q are compatible and there is $r \leq p$ incompatible with q . $S \subseteq \mathbb{P}$ is a *splitting family* if every member of \mathbb{P} is split by a member of S . The *splitting number* $\mathfrak{s}(\mathbb{P})$ of \mathbb{P} is the least size of a splitting family. Unlike \mathfrak{h} , \mathfrak{t} and \mathfrak{s} are not invariant under forcing equivalence: e.g. $\mathfrak{t}(\mathbb{A}) = \aleph_0$ for every complete atomless Boolean algebra \mathbb{A} . Also the base-matrix theorem [BPS] (see also [BS, Theorem 3.4]) implies that $\mathfrak{s}(\text{r.o.}(\mathcal{P}(\omega)/\text{fin})) = \mathfrak{h} := \mathfrak{h}(\mathcal{P}(\omega)/\text{fin})$ while $\mathfrak{h} < \mathfrak{s} := \mathfrak{s}(\mathcal{P}(\omega)/\text{fin})$ is consistent (see below for $\mathcal{P}(\omega)/\text{fin}$).

Fact 3. $\mathfrak{t}(\mathbb{P})$ is a regular cardinal. \square

Fact 4. $\mathfrak{t}(\mathbb{P}) \leq \mathfrak{h}(\mathbb{P}) \leq \mathfrak{s}(\mathbb{P})$.

Proof. Let D_α , $\alpha < \mathfrak{h}(\mathbb{P})$, be open dense such that there is $p \in \mathbb{P}$ with $\bigcap_{\alpha < \mathfrak{h}(\mathbb{P})} D_\alpha \cap O_p = \emptyset$. Recursively construct $p_\alpha \in D_\alpha \cap O_p$ such that $p_\alpha \geq p_\beta$ for $\alpha \leq \beta$. If there is a limit $\lambda < \mathfrak{h}(\mathbb{P})$ such that p_λ cannot be found, $\{p_\alpha : \alpha < \lambda\}$ is a tower and $\mathfrak{t}(\mathbb{P}) \leq cf(\lambda)$. Otherwise $\{p_\alpha : \alpha < \mathfrak{h}(\mathbb{P})\}$ must be a tower.

Let $\{p_\alpha : \alpha < \mathfrak{s}(\mathbb{P})\}$ be a splitting family. For each α let $A_\alpha \subseteq \mathbb{P}$ be a maximal antichain containing p_α . Clearly the A_α have no common refinement. \square

Thus $\mathfrak{t}(\mathbb{P})$ and $\mathfrak{s}(\mathbb{P})$ are useful because they give natural lower and upper bounds of $\mathfrak{h}(\mathbb{P})$, respectively.

2 Products and reduced powers

For separative partial orders \mathbb{P} and \mathbb{Q} consider the *product* $\mathbb{P} \times \mathbb{Q}$ equipped with the product ordering (that is, $(p', q') \leq_{\mathbb{P} \times \mathbb{Q}} (p, q)$ iff $p' \leq_{\mathbb{P}} p$ and $q' \leq_{\mathbb{Q}} q$). Since $\mathbb{P} <_{\circ} \mathbb{P} \times \mathbb{Q}$ and $\mathbb{Q} <_{\circ} \mathbb{P} \times \mathbb{Q}$ we see

Fact 5. $\mathfrak{h}(\mathbb{P} \times \mathbb{Q}) \leq \min\{\mathfrak{h}(\mathbb{P}), \mathfrak{h}(\mathbb{Q})\}$. \square

Notice that if \mathbb{A} and \mathbb{B} are Boolean algebras, then $\mathbb{A} \times \mathbb{B}$ denotes what is called the *free product* in Boolean algebra theory, namely, $(\mathbb{A}^+ \times \mathbb{B}^+) \cup \{0\}$.

For a Boolean algebra \mathbb{A} let $\mathbb{A}^\omega/\text{fin} := \{[f] : f \in \mathbb{A}^\omega\}$ where $[f] = \{g \in \mathbb{A}^\omega : \forall^\infty n (f(n) = g(n))\}$, ordered by $[f] \leq [g]$ if $f(n) \leq g(n)$ holds for almost all n . The *reduced power* $\mathbb{A}^\omega/\text{fin}$ is again a Boolean algebra.

Fact 6. If $\mathbb{A} <_{\circ} \mathbb{B}$ then $\mathbb{A}^\omega/\text{fin} <_{\circ} \mathbb{B}^\omega/\text{fin}$ (and thus $\mathfrak{h}(\mathbb{A}^\omega/\text{fin}) \geq \mathfrak{h}(\mathbb{B}^\omega/\text{fin})$). \square

If \mathbb{A} is the trivial algebra $\{0, 1\}$, we see $\mathbb{A}^\omega/\text{fin} \cong \mathcal{P}(\omega)/\text{fin}$ where $\mathcal{P}(\omega)/\text{fin} := \{[A] : A \subseteq \omega\}$ with $[A] = \{B \subseteq \omega : |A \Delta B| < \aleph_0\}$, ordered by $[A] \leq [B]$ if $|A \setminus B| < \aleph_0$. In particular $\mathfrak{h}(\mathbb{B}^\omega/\text{fin}) \leq \mathfrak{h}$ for any Boolean algebra \mathbb{B} where $\mathfrak{h} := \mathfrak{h}(\mathcal{P}(\omega)/\text{fin})$.

Stone-Čech remainders. Much of the interest in Boolean algebras of the form $\mathbb{A}^\omega/\text{fin}$ stems from the fact their completion is isomorphic to the regular open algebra $\text{r.o.}(X^*)$ of the *Stone-Čech remainder* X^* of some natural space X . Briefly recall the construction of the *Stone-Čech compactification* βX of a normal space X [En, Section 3.6]. Let βX be the family of all ultrafilters of closed subsets of X . Identify $x \in X$ with $\mathcal{U}(x) = \{A \subseteq X \text{ closed} : x \in A\} \in \beta X$. Clearly $\bigcap \mathcal{U}(x) = \{x\}$. In fact, the maximality of any $\mathcal{U} \in \beta X$ entails that either $\bigcap \mathcal{U} = \{x\}$ for some x and then $\mathcal{U} = \mathcal{U}(x)$ or $\bigcap \mathcal{U} = \emptyset$ and \mathcal{U} is a *free ultrafilter*. Thus $X^* = \beta X \setminus X$ is the space of free ultrafilters of closed sets. For $O \subseteq X$ open let $O^* = \{\mathcal{U} \in \beta X : \exists A \in \mathcal{U} \text{ with } A \subseteq O\}$. Clearly $O^* \cap X = O$. The sets O^* , $O \subseteq X$ open, form a basis of the topology of βX and thus the $O^* \cap X^*$ are a basis of the topology of X^* .

We come to specific examples. First let $X = \omega$, equipped with the discrete topology. $\beta\omega$ is the space of all ultrafilters on ω and ω^* is the space of free ultrafilters. Basic open sets are of the form $O^* = \{\mathcal{U} \in \beta\omega : O \in \mathcal{U}\}$ for $O \subseteq \omega$ and, in fact, every regular open set is of this form so that $\text{r.o.}(\beta\omega) = \mathcal{P}(\omega)$. Basic non-empty open sets of ω^* are of the form $O^* \cap \omega^*$ where $O \subseteq \omega$ is infinite. If $|O_0 \Delta O_1| < \aleph_0$, then clearly $O_0^* \cap \omega^* = O_1^* \cap \omega^*$. Thus a dense subset of $\text{r.o.}(\omega^*)^+$ is isomorphic to $\mathcal{P}(\omega)/\text{fin}^+$ and we obtain

Fact 7. $\text{r.o.}(\omega^*) = \text{r.o.}(\mathcal{P}(\omega)/\text{fin})$. \square

Similarly, we get $\text{r.o.}(\beta\omega \times \beta\omega) = \text{r.o.}(\mathcal{P}(\omega) \times \mathcal{P}(\omega))$ and $\text{r.o.}(\omega^* \times \omega^*) = \text{r.o.}(\mathcal{P}(\omega)/\text{fin} \times \mathcal{P}(\omega)/\text{fin})$ etc.

Next, let $X = \mathbb{R}$, equipped with the standard topology. For $s \in 2^{<\omega}$, $n \in \mathbb{Z}$ and $\epsilon > 0$, let

$$O_{s,n,\epsilon} = (n + \sum \{ \frac{1}{2^{i+1}} : i < |s| \text{ and } s(i) = 1 \} - \epsilon, \\ n + \sum \{ \frac{1}{2^{i+1}} : i < |s| \text{ and } s(i) = 1 \} + \frac{1}{2^{|s|}} + \epsilon)$$

Clearly the $O_{s,n,\epsilon}$ are a basis of the topology of \mathbb{R} and so the $O_{s,n,\epsilon}^*$ are a basis of the topology of $\beta\mathbb{R}$. For every infinite partial function $f : \mathbb{Z} \rightarrow 2^{<\omega}$ let

$$O_f = \bigcup_{n \in \text{dom}(f)} O_{f(n),n,\epsilon_n}$$

where $\epsilon_n = \min\{\frac{1}{2^{f(i)+5}} : n-1 \leq i \leq n+1\}$ and notice that the $O_f^* \cap \mathbb{R}^*$ form a basis of regular open sets of the topology of \mathbb{R}^* (indeed every $\mathcal{U} \in \mathbb{R}^*$ contains only unbounded closed sets; otherwise $\bigcap \mathcal{U} \neq \emptyset$ by compactness; thus for bounded $O \subseteq \mathbb{R}$, $O^* \cap \mathbb{R}^* = \emptyset$, and it is easy to see every unbounded $O \subseteq \mathbb{R}$ contains a set of the form O_f). If $\text{dom}(f) =^* \text{dom}(g)$ and $f(n) = g(n)$ for almost all $n \in \text{dom}(f)$ then $O_f^* \cap \mathbb{R}^* = O_g^* \cap \mathbb{R}^*$. Otherwise they are distinct (by choice of the ϵ_n). This means that $\text{r.o.}(\mathbb{R}^*)^+$ has a dense subset isomorphic to $F/\text{fin} := \{[f] : f \in F\}$ where $F = \{f : \mathbb{Z} \rightarrow 2^{<\omega} : \text{dom}(f) \text{ is infinite}\}$ and $[f] = \{g \in F : \text{dom}(g) =^* \text{dom}(f) \text{ and } \forall^\infty n \in \text{dom}(g) (g(n) = f(n))\}$, ordered by $[f] \leq [g]$ if $\text{dom}(f) \subseteq^* \text{dom}(g)$ and $f(n) \supseteq g(n)$ holds for almost all $n \in \text{dom}(f)$.

Let \mathbb{C} be *Cohen forcing*, that is, the algebra of clopen subsets of the Cantor space 2^ω , ordered by inclusion. Since $\{[s] : s \in 2^{<\omega}\}$ is a dense subset of \mathbb{C}^+ , \mathbb{C}^+ has a dense subset isomorphic to $2^{<\omega}$ ordered by reverse inclusion. Thus $\mathbb{C}^\omega/\text{fin}^+$ has a dense subset isomorphic to F/fin . This shows

Fact 8. $\text{r.o.}(\mathbb{R}^*) = \text{r.o.}(\mathbb{C}^\omega/\text{fin})$. \square

The above discussion motivates the investigation of cardinal numbers like $\mathfrak{h} = \mathfrak{h}(\text{r.o.}(\omega^*))$, $\mathfrak{h}_2 := \mathfrak{h}(\mathcal{P}(\omega)/\text{fin} \times \mathcal{P}(\omega)/\text{fin}) = \mathfrak{h}(\text{r.o.}(\omega^* \times \omega^*))$, $\mathfrak{h}(\mathbb{C}^\omega/\text{fin}) = \mathfrak{h}(\text{r.o.}(\mathbb{R}^*))$ etc.

Distributivity numbers of products and reduced powers. We know already $\mathfrak{h}_2 \leq \mathfrak{h}$. The following is easy to see

Fact 9. $\mathfrak{t} \leq \mathfrak{h}_2$. \square

On the other hand \mathfrak{h}_2 may be less than \mathfrak{h} .

Theorem 1. (Shelah-Spinas [SS1]) $\text{CON}(\mathfrak{h}_2 < \mathfrak{h})$.

In fact $\mathfrak{h}_2 < \mathfrak{h}$ holds in the iterated Mathias model (the ω_2 -stage countable support iteration of Mathias forcing over a model of CH). \square

In fact Shelah and Spinus also obtained the consistency of $\mathfrak{h}_{n+1} < \mathfrak{h}_n$ for any n [SS2] where $\mathfrak{h}_n := \mathfrak{h}((\mathcal{P}(\omega)/\text{fin})^n)$. Since $\mathfrak{t} \leq \mathfrak{h}_n$ for all n , the consistency of $\mathfrak{t} < \mathfrak{h}_n$ follows immediately.

We know already $\mathfrak{h}(\mathbb{C}^\omega/\text{fin}) \leq \mathfrak{h}$. Again, the inequality is consistently strict.

Theorem 2. (Dow [Do]) $CON(\mathfrak{h}(\mathbb{C}^\omega/\text{fin}) < \mathfrak{h})$.

In fact $\mathfrak{h}(\mathbb{C}^\omega/\text{fin}) < \mathfrak{h}$ holds in the iterated Mathias model. \square

The similarity to the Shelah-Spinas result lead to the following

Question 1. (Dow [Do]) Is $\mathfrak{h}(\mathbb{C}^\omega/\text{fin} \times \mathbb{C}^\omega/\text{fin}) = \mathfrak{h}(\mathbb{C}^\omega/\text{fin})$? Is $\mathfrak{h}(\mathbb{C}^\omega/\text{fin}) \leq \mathfrak{h}_2$?

Note that $\mathfrak{h}(\mathbb{C}^\omega/\text{fin} \times \mathbb{C}^\omega/\text{fin}) \leq \mathfrak{h}_2$ because $\mathbb{C}^\omega/\text{fin} \times \mathbb{C}^\omega/\text{fin} \leq \mathcal{P}(\omega)/\text{fin} \times \mathcal{P}(\omega)/\text{fin}$. Upper and lower bounds for $\mathfrak{h}(\mathbb{C}^\omega/\text{fin})$ are given by

Theorem 3. (Balcar-Hrušák [BH]) $\mathfrak{t} \leq \mathfrak{h}(\mathbb{C}^\omega/\text{fin}) \leq \text{add}(\mathcal{M})$. \square

Here $\text{add}(\mathcal{M})$ denotes the *additivity of the meager ideal*, that is, the least size of a family of meager sets whose union is not meager.

Balcar and Hrušák also observed [BH] that $\mathfrak{t} < \mathfrak{h}(\mathbb{C}^\omega/\text{fin})$ is consistent. Moreover, Dow's Theorem 2 is a Corollary of Theorem 3. Namely, it is well-known (and much easier to prove than Dow's argument for $\mathfrak{h}(\mathbb{C}^\omega/\text{fin}) = \aleph_1$) that $\text{add}(\mathcal{M}) = \aleph_1$ in the iterated Mathias model (see, e.g., [BJ]). On the other hand, since $\mathfrak{h} < \text{add}(\mathcal{M})$ in the Hechler model, the consistency of $\mathfrak{h}(\mathbb{C}^\omega/\text{fin}) < \text{add}(\mathcal{M})$ follows. This naturally leads to

Question 2. (Balcar-Hrušák [BH]) Is $\mathfrak{h}(\mathbb{C}^\omega/\text{fin}) < \min\{\mathfrak{h}, \text{add}(\mathcal{M})\}$ consistent?

Both questions can be answered with basically the same method.

Theorem 4. [Br3] $CON(\mathfrak{h}(\mathbb{C}^\omega/\text{fin}) < \min\{\mathfrak{h}, \text{add}(\mathcal{M})\})$.

Theorem 5. [Br3] $CON(\mathfrak{h}_2 < \mathfrak{h}(\mathbb{C}^\omega/\text{fin}))$.

Thus $\mathfrak{h}(\mathbb{C}^\omega/\text{fin} \times \mathbb{C}^\omega/\text{fin}) < \mathfrak{h}(\mathbb{C}^\omega/\text{fin})$ is consistent as well.

Notice that the converse, namely, the consistency of $\mathfrak{h}_2 > \mathfrak{h}(\mathbb{C}^\omega/\text{fin})$, follows from the consistency of $\mathfrak{h}_2 > \text{add}(\mathcal{M})$ established by Shelah and Spinus [SS2] and from Theorem 3.

Sketch of proof. We briefly sketch the proof of Theorem 4. Unlike earlier results on the independence of distributivity numbers ([Do], [SS1], [SS2]), we use a finite support iteration $\langle \mathbb{P}_\alpha, \dot{Q}_\alpha : \alpha < \omega_2 \rangle$ of ccc forcing over a model of CH . This is natural because we have to add Cohen reals anyway (in Theorem 4 we want to add increase $\text{add}(\mathcal{M})$, and in Theorem 5 we have to increase $\text{add}(\mathcal{M})$ by Theorem 3).

Roughly speaking, the iteration adds dominating reals (via Hechler forcing \mathbb{D} , see [BJ]) in successor stages and limit stages of cofinality ω while we use *Laver forcing* $\mathbb{L}_\mathcal{U}$ with a Ramsey ultrafilter \mathcal{U} at limit stages of cofinality ω_1 (see below for the definition). The Hechler reals (as well as the Laver reals) guarantee that $\mathfrak{b} = \aleph_2$ while the Cohen reals give $\text{cov}(\mathcal{M}) = \aleph_2$. So $\text{add}(\mathcal{M}) = \min\{\mathfrak{b}, \text{cov}(\mathcal{M})\} = \aleph_2$ holds.

Recall that an ultrafilter \mathcal{U} on ω is *Ramsey* if for all partitions $\langle X_n : n \in \omega \rangle$ of ω either $X_n \in \mathcal{U}$ for some $n \in \omega$ or there is $Y \in \mathcal{U}$ with $|Y \cap X_n| \leq 1$ for all n . $\mathbb{L}_\mathcal{U}$ consists of all trees $T \subseteq \omega^{<\omega}$ such that for all $s \in T$ below the

stem (i.e. $s \supseteq \text{stem}(T)$) the set of successor nodes $\{n : s \frown n \in T\}$ belongs to \mathcal{U} . $\mathbb{L}_{\mathcal{U}}$ is ordered by inclusion. It is easy to see that the generic *Laver real* $\ell_{\mathcal{U}} := \bigcap \{[T] : T \in G\} \in \omega^\omega$ dominates the ground model reals and that $\text{ran}(\ell_{\mathcal{U}}) \subseteq \omega$ diagonalizes \mathcal{U} (i.e. $\ell_{\mathcal{U}} \subseteq^* U$ for all $U \in \mathcal{U}$). Here G denotes the generic filter. Thus iterating $\mathbb{L}_{\mathcal{U}}$ naturally increases \mathfrak{b} and \mathfrak{s} . The effect of $\mathbb{L}_{\mathcal{U}}$ on \mathfrak{h} , however, is a more subtle issue and depends very much on the choice of the ultrafilter \mathcal{U} . In some situations \mathfrak{h} (and its relatives) stay small (see [Br2, Section 2] for such a construction) and we obtain a natural model for $\mathfrak{h} < \min\{\mathfrak{b}, \mathfrak{s}\}$, the consistency of which was originally obtained by Shelah [Sh1] (see also [Sh2, Theorem VI.8.2]).

We assume $\diamond_{S_1^2}$ holds in the ground model. This means there is a sequence $\langle Z_\alpha : cf(\alpha) = \omega_1, \alpha < \omega_2 \rangle$ such that for all $Z \subseteq \omega_2$, the set $\{\alpha < \omega_2 : cf(\alpha) = \omega_1 \text{ and } Z \cap \alpha = Z_\alpha\}$ is stationary. $\diamond_{S_1^2}$ is used for guessing (initial segments of) names for potential witnesses for $\mathfrak{h} = \aleph_1$. Notice that if \dot{A} is a \mathbb{P}_{ω_2} -name for such a witness, then by *CH* and *ccc*, \dot{A} can be thought of as an object of size ω_2 and can be coded into a subset of ω_2 . Similarly if $\alpha < \omega_2$, $|\alpha| = \aleph_1$, and \dot{A} is a \mathbb{P}_α -name for a witness of $\mathfrak{h} = \aleph_1$, \dot{A} can be coded into a subset of α . Thus, if at stage α where $cf(\alpha) = \omega_1$, Z_α codes such a \mathbb{P}_α -name $\dot{A} = \{\dot{A}_\beta : \beta < \omega_1\}$, then we construct a Ramsey ultrafilter \mathcal{U}_α such that $\Vdash_\alpha \dot{U}_\alpha \cap \dot{A}_\beta \neq \emptyset$ for all $\beta < \omega_1$ and force with $\mathbb{Q}_\alpha = \mathbb{L}_{\mathcal{U}_\alpha}$. This destroys the witness \dot{A} . Now, if \dot{A} is a \mathbb{P}_{ω_2} -name for a witness for $\mathfrak{h} = \aleph_1$ coded by $Z \subseteq \omega_2$, then $C = \{\alpha < \omega_2 : cf(\alpha) = \omega_1 \text{ and } \dot{A} \restriction \alpha \text{ is a } \mathbb{P}_\alpha\text{-name for a witness for } \mathfrak{h} = \aleph_1\}$ is ω_1 -club. By $\diamond_{S_1^2}$ there is $\alpha \in C$ with $Z \cap \alpha = Z_\alpha$. So Z_α codes $\dot{A} \restriction \alpha$ and $\mathbb{L}_{\mathcal{U}_\alpha}$ destroys $\dot{A} \restriction \alpha$ and also \dot{A} . This shows $\mathfrak{h} = \aleph_2$.

The most difficult part of the argument is the proof of $\mathfrak{h}(\mathbb{C}^\omega/\text{fin}) = \aleph_1$. We build a witness $\mathcal{F} = \{F_\beta \subseteq \mathbb{C}^\omega : \beta < \omega_1\}$ along the iteration. The main point is that if the Ramsey ultrafilter \mathcal{U}_α is carefully chosen, then $\mathbb{Q}_\alpha = \mathbb{L}_{\mathcal{U}_\alpha}$ does not destroy (the initial segment of) this witness \mathcal{F} . This is a technical argument which relies heavily on a *rank analysis* of $\mathbb{L}_{\mathcal{U}_\alpha}$ -names. See [Br3] for details. To be able to build the required Ramsey ultrafilter in limit stages of cofinality ω_1 , we use the Hechler reals which we added in successor stages. For Hechler forcing it is much easier to see that it preserves (the initial segment of) the witness \mathcal{F} . Thus $\mathfrak{h}(\mathbb{C}^\omega/\text{fin}) = \aleph_1$ follows. \square

We close this section with some comments and questions on related cardinals. Balcar and Hrušák [BH] proved that $\mathfrak{t}(\mathbb{C}^\omega/\text{fin}) = \mathfrak{t}$ (and thus $\mathfrak{t}(\mathbb{C}^\omega/\text{fin}) < \mathfrak{h}(\mathbb{C}^\omega/\text{fin})$ is consistent as well, see above). But little seems to be known about $\mathfrak{s}(\mathbb{C}^\omega/\text{fin})$ except for the trivial $\mathfrak{s}(\mathbb{C}^\omega/\text{fin}) \leq \mathfrak{s}$.

Problem 1. Investigate $\mathfrak{s}(\mathbb{C}^\omega/\text{fin})$! Investigate $\mathfrak{x}(\mathbb{C}^\omega/\text{fin})$ for other cardinal invariants \mathfrak{x} !

For a topological space X without isolated points, the *Baire number* of X (also called *Novák number*), $\mathfrak{n}(X)$, is the least size of a family of nowhere dense sets covering X . Let $\mathfrak{n} := \mathfrak{n}(\omega^*)$.

Theorem 6. (Balcar-Pelant-Simon [BPS], see also [BS, Theorem 3.10])

(i) If $\mathfrak{h} < \mathfrak{c}$, then $\mathfrak{h} \leq \mathfrak{n} \leq \mathfrak{h}^+$.

(ii) If $\mathfrak{h} = \mathfrak{c}$, then $\mathfrak{c} \leq \mathfrak{n} \leq 2^{\mathfrak{c}}$. \square

The analogous result holds for $\mathfrak{h}(\mathbb{R}^*)$ and $\mathfrak{n}(\mathbb{R}^*)$. Also $\mathfrak{n}(\mathbb{R}^*) \leq \mathfrak{n}$ is easy to see, but the following is still open.

Question 3. (van Douwen, see [Do]) Is $\mathfrak{n}(\mathbb{R}^*) < \mathfrak{n}$ consistent?

3 Further friends of $\mathcal{P}(\omega)/\text{fin}$

We briefly discuss the distributivity number of other structures related to $\mathcal{P}(\omega)/\text{fin}$.

Dense (\mathbb{Q}) / nwd. Let $\text{Dense}(\mathbb{Q})$ denote the family of dense subsets of the rationals \mathbb{Q} , and let nwd stand for the nowhere dense sets of rationals. Let $\text{Dense}(\mathbb{Q})/\text{nwd} = \{[A] : A \in \text{Dense}(\mathbb{Q})\}$ where $[A] = \{B : A \Delta B \in \text{nwd}\}$ for $A \in \text{Dense}(\mathbb{Q})$, ordered by $[A] \leq [B]$ if $A \setminus B \in \text{nwd}$. Let $\mathfrak{h}_{\mathbb{Q}} = \mathfrak{h}(\text{Dense}(\mathbb{Q})/\text{nwd})$. $\mathfrak{s}_{\mathbb{Q}}$ and $\mathfrak{t}_{\mathbb{Q}}$ are defined similarly. The investigation of $\text{Dense}(\mathbb{Q})/\text{nwd}$ has been started by Balcar, Hernández and Hrušák [BHH].

Theorem 7. (Balcar-Hernández-Hrušák [BHH], Brendle [Br2])

(i) $\mathfrak{t}_{\mathbb{Q}} = \mathfrak{t}$.

(ii) $\mathfrak{s}_{\mathbb{Q}} \leq \min\{\mathfrak{s}, \text{add}(\mathcal{M})\}$. \square

Balcar, Hernández and Hrušák [BHH] also proved the consistency of $\mathfrak{t}_{\mathbb{Q}} < \mathfrak{h}_{\mathbb{Q}}$ and of $\mathfrak{h}_{\mathbb{Q}} < \mathfrak{h}$. In fact, by (ii) of Theorem 7, $\mathfrak{s}_{\mathbb{Q}} < \mathfrak{h}$ holds in the iterated Mathias model. Furthermore:

Theorem 8. [Br2]

(i) $\text{CON}(\mathfrak{h}_{\mathbb{Q}} < \mathfrak{s}_{\mathbb{Q}})$.

(ii) $\text{CON}(\mathfrak{h} < \mathfrak{h}_{\mathbb{Q}})$. \square

The argument for the proof of (ii) is similar to the argument for Theorems 4 and 5, see above. The following is still open.

Question 4. [Br2] Is $\mathfrak{s}_{\mathbb{Q}} < \min\{\mathfrak{s}, \text{add}(\mathcal{M})\}$ consistent?

Partitions of ω . Let (ω) denote the collection of partitions of ω . $(\omega)^{\omega}$ is the *infinite partitions* of ω (i.e. the partitions into infinitely many blocks), and $(\omega)^c$ is the *non-trivial partitions* of ω . Here, we say $A \in (\omega)$ is trivial if $\{n\} \in A$ for almost all n (equivalently, A has no infinite block and almost all blocks are singletons). Write $A \leq B$ if A is *coarser* than B iff all blocks of A are unions of blocks of B . Say X is a *finite coarsening* of A if X is gotten from A by merging finitely many blocks of A . Write $A \leq^* B$ if there is a finite coarsening X of A such that $X \leq B$. Say $A =^* B$ if $A \leq^* B$ and $B \leq^* A$ iff there is

X which is a finite coarsening of both A and B . Let $[A] = \{B : A \leq^* B\}$ and set $[A] \leq [B]$ if $A \leq^* B$. $((\omega)^\omega / \leq^*, \leq)$ is the separative quotient of $((\omega)^\omega, \leq)$. It is called the *dual structure* and we let $\mathfrak{h}_d = \mathfrak{h}((\omega)^\omega / \leq^*)$ etc. As usual, we work with $((\omega)^\omega, \leq^*)$ instead of $((\omega)^\omega / \leq^*, \leq)$. It is easy to see that $\mathcal{P}(\omega)/\text{fin} < (\omega)^\omega / \leq^*$; namely, $h : (\omega)^\omega \rightarrow [\omega]^\omega$ given by $h(A) = \{\min(b) : b \in A\}$ induces the projection mapping giving rise to the complete embedding. Thus $\mathfrak{h}_d \leq \mathfrak{h}$. The investigation of cardinal invariants of $((\omega)^\omega, \leq^*)$ has been started by Cichoń, Krawczyk, Majcher-Iwanow and Węglorz [CKMW].

Theorem 9. (Carlson [Mat]) $\mathfrak{t}_d = \aleph_1$. \square

Theorem 10. (i) (Halbeisen [Ha]) $\text{CON}(\mathfrak{h}_d > \aleph_1)$.

Namely, $\mathfrak{c} = \mathfrak{h}_d = \aleph_2$ holds in the iterated dual Mathias model.

(ii) (Spinas [Sp]) $\text{CON}(\mathfrak{h}_d < \mathfrak{h})$.

In fact $\mathfrak{h}_d < \mathfrak{h}$ holds in the iterated Mathias model.

(iii) [Br1] $\text{CON}(\mathfrak{h}_d = \aleph_1 + MA + \neg CH)$. \square

Note that (iii) strengthens (ii) because MA implies $\mathfrak{t} = \mathfrak{c}$ and, thus, $\mathfrak{h} = \mathfrak{c}$ and $\mathfrak{h}(\mathbb{P}) = \mathfrak{c}$ where \mathbb{P} is any of the partial orders considered in Section 2 or $\mathbb{P} = \text{Dense}(\mathbb{Q})/\text{nwd}$. (The main distinction seems to be that for all \mathbb{P} considered earlier in this paper, $\mathfrak{t}(\mathbb{P}) = \mathfrak{t}$ in ZFC while $\mathfrak{t}_d = \aleph_1$.) On the other hand, Cichoń et al. [CKMW] already observed that MA implies $\mathfrak{s}_d = \mathfrak{c}$ so that $\mathfrak{h}_d < \mathfrak{s}_d$ is consistent as well.

There is another natural structure associated with (ω) , which is obtained by turning the order upside down and looking at refining instead of coarsening. Say $A \leq_c B$ if A is *finer* than B iff $B \leq A$. X is a *finite refinement* of A if for some finite $x \subseteq \omega$, $X = \{b \setminus x : b \in A\} \cup \{\{n\} : n \in x\}$. Write $A \leq_c^* B$ if there is a finite refinement X of A such that $X \leq_c B$. Say $A =_c^* B$ if $A \leq_c^* B$ and $B \leq_c^* A$ iff there is X which is a finite refinement of both A and B . Notice that $A \leq_c^* B$ implies $B \leq^* A$ (and equivalence holds for partitions which contain only finite blocks). As usual let $[A] = \{B : A =_c^* B\}$, $[A] \leq [B]$ if $A \leq_c^* B$ and consider the *converse dual structure* $((\omega)^c / \leq_c^*, \leq)$ which may be identified with $((\omega)^c, \leq_c^*)$. The reason for considering \leq_c^* instead of \geq^* is that the former gives indeed rise to the separative quotient of $((\omega)^c, \leq_c)$ (while $((\omega)^c, \geq^*)$ does not). This structure has been investigated by Majcher-Iwanow [Maj]. Again $\mathcal{P}(\omega)/\text{fin} < (\omega)^c / \leq_c^*$; but more is true: $((\omega)^c, \leq_c^*)$ is locally isomorphic to $([\omega]^\omega, \subseteq^*)$ [Maj] so that $\text{r.o.}(\mathcal{P}(\omega)/\text{fin}) = \text{r.o.}((\omega)^c / \leq_c^*)$. Thus $\mathfrak{h}_c = \mathfrak{h}$ where $\mathfrak{h}_c = \mathfrak{h}((\omega)^c / \leq_c^*)$. In fact, equality also holds for several other cardinal invariants of the continuum; e.g., $\mathfrak{t}_c = \mathfrak{t}$ and $\mathfrak{s}_c = \mathfrak{s}$, see [BZ] for details.

The General Philosophy behind the results obtained so far is that distributivity numbers are independent unless there is an order relationship for trivial reasons, namely, unless there is a complete embedding between the partial orderings. Indeed, in all cases investigated so far, either $\mathbb{P} < \mathbb{Q}$ or $\text{CON}(\mathfrak{h}(\mathbb{P}) < \mathfrak{h}(\mathbb{Q}))$ has been established.

References

- [BHH] B. Balcar, F. Hernández-Hernández and M. Hrušák, *Combinatorics of dense subsets of the rationals*, Fund. Math. 183 (2004), 59-80.
- [BH] B. Balcar and M. Hrušák, *Distributivity of the algebra of regular open subsets of $\beta\mathbb{R} \setminus \mathbb{R}$* , Top. Appl.
- [BPS] B. Balcar, J. Pelant and P. Simon, *The space of ultrafilters on \mathbb{N} covered by nowhere dense sets*, Fund. Math. 110 (1980), 11-24.
- [BS] B. Balcar and P. Simon, *Disjoint refinement*, in: Handbook of Boolean algebra (J.D. Monk and R. Bonnet, eds.), North-Holland, Amsterdam (1989), 335-386.
- [BJ] T. Bartoszyński and H. Judah, *Set Theory, On the structure of the real line*, A K Peters, Wellesley, 1995.
- [Br1] J. Brendle, *Martin's Axiom and the dual distributivity number*, Math. Log. Quart. 46 (2000), 241-248.
- [Br2] J. Brendle, *Van Douwen's diagram for dense sets of rationals*, Ann. Pure Appl. Logic.
- [Br3] J. Brendle, *Independence for distributivity numbers*, preprint.
- [BZ] J. Brendle and S. Zhang, *Converse dual cardinals*, J. Symbolic Logic.
- [CKMW] J. Cichoń, A. Krawczyk, B. Majcher-Iwanow, B. Węglorz, *Dualization of the van Douwen Diagram*, J. Symbolic Logic 65 (2000), 959-968.
- [Do] A. Dow, *The regular open algebra of $\beta\mathbb{R} \setminus \mathbb{R}$ is not equal to the completion of $\mathcal{P}(\omega)/\text{fin}$* , Fund. Math. 157 (1998), 33-41.
- [En] R. Engelking, *General Topology, Revised and completed edition*, Heldermann, Berlin, 1989.
- [Ha] L. Halbeisen, *On shattering, splitting and reaping partitions*, Math. Log. Quart. 44 (1998), 123-134.
- [Maj] B. Majcher-Iwanow, *Cardinal invariants of the lattice of partitions*, Comment. Math. Univ. Carolinae 41 (2000), 543-558.
- [Mat] P. Matet, *Partitions and filters*, J. Symbolic Logic 51 (1986), 12-21.
- [Sh1] S. Shelah, *On cardinal invariants of the continuum*, in: Axiomatic set theory (J. Baumgartner et al., eds.) Contemp. Math. 31 (1984), Amer. Math. Soc., Providence, 183-207.
- [Sh2] S. Shelah, *Proper and Improper Forcing*, Springer, Berlin, 1998.

- [SS1] S. Shelah and O. Spinas, *The distributivity number of $\mathcal{P}(\omega)/\text{fin}$ and its square*, Trans. Amer. Math. Soc. 352 (2000), 2023-2047.
- [SS2] S. Shelah and O. Spinas, *The distributivity numbers of finite products of $\mathcal{P}(\omega)/\text{fin}$* , Fund. Math. 158 (1998), 81-93.
- [Sp] O. Spinas, *Partition numbers*, Ann. Pure Appl. Logic 90 (1997), 243-262.